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Investigation of Elementary Vibrations: Derivation, Experimental Analysis, and Key Findings

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Abstract

This report includes a detailed discussion on the results obtained through the Elementary Vibrations experiment. At the beginning of the report, the purpose and the background of the experiment will be discussed and the two equations to calculate the natural frequency of free vibrations will be derived. Then the apparatus used and the experimental procedure will be described and the results recorded during the experiment will be shown and discussed in detail. At the end, the key findings will be explained.

Keywords: Elementary vibrations, natural frequency, free vibrations, experimental procedure, apparatus, results analysis, key findings, and vibration dynamics.

INTRODUCTION

The Elementary Vibrations experiment was performed in order to calculate the theoretical and practical natural frequencies of a spring under different condition (certain assumptions were made) [1]. The experiment was performed thrice:

1) Object directly connected to the spring. (Fig.2)
2) Object connected to the spring via a small pulley in between.(Fig.3)
3) Object connected to the spring via a big pulley in between.(Fig.3)

The whole experimental procedure is explained later.

BACKGROUND

"A Simple Harmonic Motion is a special type of vibratory motion, in which acceleration is directly proportional to the displacement and the acceleration is always directed towards the mean (the position at which the object is hanging at rest when no forces are applied) position [2]. Mathematically we can represent these kinds of motion as:

$\boldsymbol{a = -\omega^2 x}$ ----- Eq.1

Where:

a = acceleration $\quad \omega$ = Angular Frequency (constant) $\quad x$ = Displacement

The negative sign in Eq.1 indicates that the acceleration is directed towards the mean position. As no external forces are applied, we can assume our motion to be simple harmonic motion.

If a force is applied to an object, the object starts oscillating until the amplitude of the oscillation die out. The object always comes to rest at its mean position, provided that the spring does not lose its elasticity [3]. The greater the distance the object covers from its mean position, greater the acceleration is and therefore it will take a greater amount of time to come at rest [4]. Another important point to note is that the spring used for the experiment is elastic. An elastic body reaches back to its original shape and size when the deforming force is removed, due to the Elastic Restoring Force (Tension Force) in the body [5]. An elastic spring obeys Hooke's law till a certain point (limit of elasticity). Hooke's law states that, "extension produced in an elastic body is directly proportional to the force applied, if the limit of elasticity is not exceeded mathematically we can write it as

$\boldsymbol{F = kx}$ ----- Eq.2

Where:

F = Force required for extension/compression   *k* = Spring Constant   *x* = Displacement

Spring constant (*k*) can be defined as, "Force per unit extension." It is a constant value for a particular spring. Greater the value of the spring constant, more the force will be required for the extension or a compression of a spring [6].

Every spring has its own limit of elasticity, i.e. the spring loses its elastic behaviour and gets deformed permanently if that limit is crossed. Fig 1 shows the total extension with the amount of force applied.

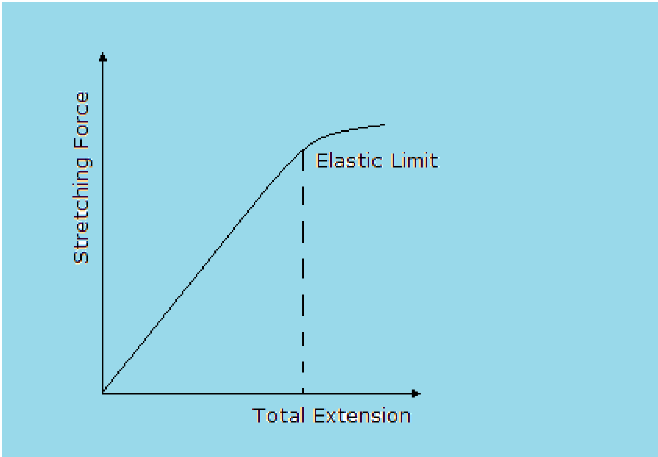

Fig 1: Force VS extension graph

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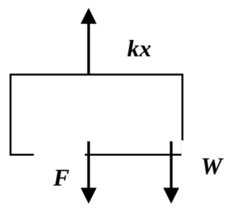
It can be seen from Fig.1 that Force is directly proportional to the extension till the limit of elasticity. Once it crosses the elastic limit, it starts to disobey Hooke's law. The area under the linear region of Fig.1 gives the Elastic Potential Energy or Strain Energy [7]. Frequency (*f*) is the number of oscillations per unit time. Mathematically it can be represented as:

$$f = \frac{1}{T} \quad \text{----- Eq.3}$$

Where *f* is the frequency and T is the Time Period for 1 oscillation. In the experiment performed, it was made sure that the extension in the spring remained within the elastic limit in order to calculate accurate natural vibrations within the spring. The derivation of the equations used to calculate the frequency of simple mass-spring system (Fig.2) and the frequency of the spring-mass-pulley system (Fig.3) are given below [8].

DERIVATIONS

Simple Mass-Spring System:



From Newton's First Law, we know:

$$F = ma \quad \text{----- Eq.4}$$

Where: ***F*** =resultant force, ***m*** = mass of the object, ***W*** = Weight, ***a*** = Acceleration

We know, acceleration is the rate of change of velocity and velocity is the rate of change of displacement "*x*".To be precise acceleration is the double derivative of the displacement:

$$a = \frac{d^2x}{dt^2} \quad \text{----- Eq.5}$$

If we consider the free body diagram for this system, neglecting the weight (W=0), we can re-write the Newton's law as:

$$F - kx = m\frac{d^2x}{dt^2} \quad \text{----- Eq.6}$$

$$F = m\frac{d^2x}{dt^2} + kx = 0 \quad \text{----- Eq.7}$$

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When the force is released, the overall force of the system becomes zero, as shown in Eq.7 and the system undergoes free vibrations [9].

The equation for the displacement of an object undergoing Simple Harmonic Motion is:

$$x = A\sin(\omega t) \quad \text{----- Eq.8}$$

Differentiating the equation gives us the velocity of the motion:

$$v = \frac{dx}{dt} = \omega A\cos(\omega t) \quad \text{----- Eq.9}$$

Differentiating the equation again gives the acceleration of the motion:

$$a = \frac{d^2x}{dt^2} = -\omega^2 A\sin(\omega t) \quad \text{----- Eq.10}$$

Substituting Eq.10 and Eq.8 into Eq.7 gives:

$$-m\omega^2 A\sin(\omega t) + kA\sin(\omega t) = 0$$

$$-m\omega^2 + k = 0$$

$$\omega = \sqrt{\frac{k}{m}} \quad \text{------Eq.11} \quad \text{from circular motion we know } T = \frac{2\pi}{\omega} \text{ ------ Eq.12}$$

Substituting Eq.11 in Eq.12 and taking the total mass as the sum of the mass of the object and one third of the mass of the spring (explained after this derivation) gives:

$$T = 2\pi\sqrt{\frac{m}{k}} \quad \text{for this system} \quad T = 2\pi\sqrt{\frac{m+m_s/3}{k}} \text{ ---- Eq.13}$$

From Eq.3, we know that the frequency "*f*" is the inverse of the time period "T":

$$f_2 = \frac{1}{2\pi}\sqrt{\frac{k}{m_o+m_s/3}} \text{ ------Eq.14 (The equation for the simple mass-spring system)}$$

Where: A = Amplitude of the oscillations m = total mass k = spring constant

ω = Angular Frequency t = total time v = Velocity x = Displacement a = acceleration

Reason for considering the mass of the spring as one third its original mass

For deducing the ratio of the mass of the spring considered for our equation, we will have to look at the Kinetic Energy of a oscillating vertical spring of length L:

$$E_k = \int_0^L \frac{1}{2} v^2 dm \text{ ------Eq.15}$$

We assume:

Uniform mass distribution and stretching in the spring takes place, that is: "$dm = \frac{m_s}{L} dz$" [3]

Velocity is the linear function of displacement from the equilibrium position of the spring, that is: "$v(z) = \frac{v_o}{L} z$"

v_o is the velocity at any instant of the point at the free end at z = L. Now considering the three assumptions and integrating Eq.15 gives: "$\boldsymbol{E_k = \frac{1}{2}\frac{m_s}{3}v_o{}^2}$" ------Eq.16

Comparing Eq.16 with the simple Kinetic Energy Equation that is "$\boldsymbol{E_k = \frac{1}{2}mv^2}$", proves that "$\boldsymbol{m = \frac{1}{3}m_s}$".

Spring-mass-pulley system

If we consider the free body diagram of the pulley from Fig.3, we get:

$2T = k(Y - l)$

Where *k(y-l)* is the spring restoring force and *T* is the tension force. Rearranging gives:

$T = \frac{1}{2}k(Y - l)$ -----Eq.17

Now consider the free body diagram of the object, which gives:

$mg = T$ ------Eq.18

By substituting Eq.17 into Eq.18, we get:

$m\ddot{X} = mg - \frac{1}{2}k(Y - l)$ -----Eq.19

Now we can calculate the total length L of the wire, by using geometry on Fig.3:

The circumference of the semi-circle on top of the pulley can be calculated by 2π, the length of the first half can be calculated by (H-Y) and the second half can be calculated by (X-Y) .

$L = (\mathrm{H} - \mathrm{Y}) + \pi r + (\mathrm{X} - \mathrm{Y}). = X - 2Y + C + \pi r$

$2\ddot{Y} = \ddot{X}$ ------Eq.20 [6]

Substituting Eq.19 into Eq.20 gives:

$2m\ddot{Y} = mg - \frac{1}{2}k(Y - l)$

This can be further solved into:

$4m\ddot{Y} + kY = 2mg + kl$

From this equation we can see that the mass m of the object is multiplied by a factor of 4, hence the Eq.14 can be written as:

$\boldsymbol{f = \frac{1}{2\pi}\sqrt{\frac{k}{4m+\frac{m_s}{3}}}}$ ---- Eq.21

APPARATUS

- Two Pulleys: Two pulleys of different mass and size were used.
- Attached Brackets: They were used to connect the pulley with the spring.
- Spring: It was connected with a cord and attached with a pulley via attached brackets. Vibrations were produced in the spring.
- Body: A body of mass 0.007 kg was used to produce vibrations in the spring.
- Light Cord: A light cord is a uniform wire with constant tension in the whole wire, which was used to connect the object with the pulley and the spring.

- Clamp: A clamp was used to fix our system on a certain height so that the object can vibrate freely without touching the horizontal surface.
- Weighing Machine: It was used to measure the mass of spring, two pulleys and the mass of object.
- Stop Watch: It was used to measure the time in which the object completes 10 oscillations

EXPERIMENTAL PROCEDURE

Schematic Diagrams:

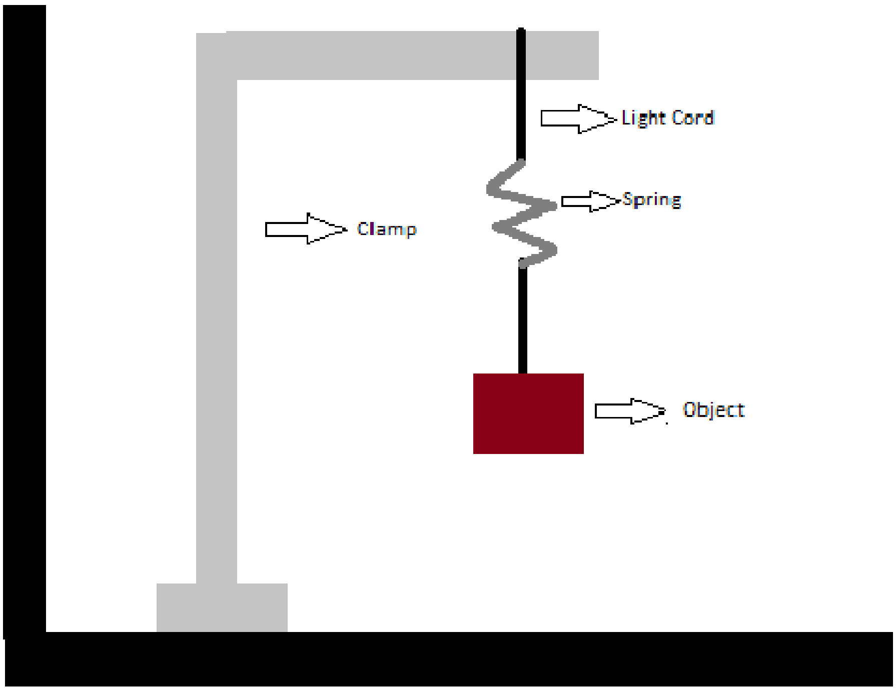
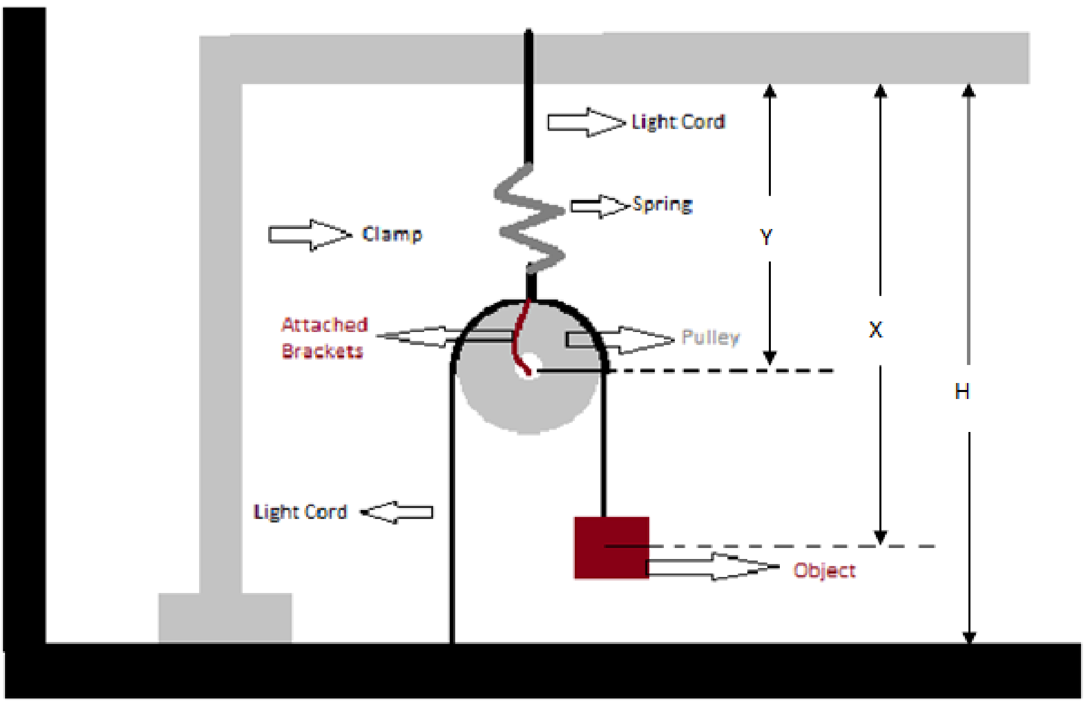


Fig.2: Object directly connected to the spring



BULLET: Jurnal Multidisiplin Ilmu
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Fig.3: Spring-Mass-Pulley System

PROCEDURE

First of all, the masses of pulleys, object and the spring were measured. Then the light cord was attached to the spring and the object of a certain mass was connected with the cord [10]. The object and the spring were then clamped at a certain height, as shown in Fig.2. After setting the system, the object was displaced downwards with enough force to create at least 10 oscillations and released [11]. We made sure that our force didn't exceed the elastic limit of the spring. The object started oscillating. The time of 10 complete oscillations was noted down. The process was repeated several times to increase accuracy [12]. The average of all the values was taken and then the time for 1 oscillation was calculated. The value of "*k*", the spring constant, was calculated from these measured values [13].

Then the system was separated and then rearranged as shown in Fig.3. The small pulley was attached to the spring via attached brackets; the rest of the system was the same. Now the same process was repeated as given in the paragraph above with this new system [14]. Using the value of *k* calculated from the previous experiment, the natural frequency of the system with small pulley is calculated. After calculating all the new values, the small pulley was replaced with the big pulley and the same procedure was repeated again [15].

RESULTS

The table below shows the values recorded during the experiment:

Body simply suspended from the spring		***Body suspended with small pulley and spring***		***Body suspended with large pulley and spring***	
Time for 10 oscillations/(s)		**Time for 10 oscillations/(s)**		**Time for 10 oscillations/(s)**	
1.	3.47	1.	6.03(out liar-ignored)	1.	8.23
2.	3.53	2.	5.97 (out liar-ignored)	2.	7.94
3.	3.87	3.	5.62	3.	8.16
4.	3.34	4.	5.62	4.	8.10
5.	3.47	5.	5.63	5.	7.81
6.	3.69	6.	5.65		
		7.	5.62		

Body simply suspended from the spring	***Body suspended with small pulley and spring***	***Body suspended with large pulley and spring***

The equation to calculate the frequency of the body simply suspended from the spring is:

$$f_2 = \frac{1}{2\pi}\sqrt{\frac{k}{m_o + m_s/3}}$$

Where:

m_o = Mass of object m_s = Mass of Spring k = Spring Constant

f_2 = Natural Frequency of the free vibrations (object simply suspended from the spring)

Now to calculate the value of k we can substitute the values recorded during the experiment:

Average Time for 10 oscillations = 3.562 s Average Time for 1 oscillations = 0.356 s Frequency for 1 oscillation= 2.808 Hz Mass of spring = 0.007 kg Mass of object = 0.352 kg	Average Time for 10 oscillations = 5.632 s Average Time for 1 oscillations = 0.563 s Frequency for 1 oscillation= 1.777 Hz Mass of spring = 0.007 kg Mass of object = 0.352 kg Mass of Small Pulley = 0.013 kg	Average Time for 10 oscillations = 8.048 s Average Time for 1 oscillations = 0.805 s Frequency for 1 oscillation= 1.243 Hz Mass of spring = 0.007 kg Mass of object = 0.352 kg Mass of Big Pulley = 0.270 kg

$$2.808 = \frac{1}{2\pi}\sqrt{\frac{k}{0.352 + 0.007/3}}$$

k = 110.8 Nm^{-1}

Using the value of k we can now calculate the theoretical value of frequency for both small and big pulley.

$$\boldsymbol{f_1 = \frac{1}{2\pi}\sqrt{\frac{k}{4m_o + m_s/3}}}$$

Where:

m_o = Mass of object m_s = Mass of Spring k = Spring Constant

f_2 = Natural Frequency of free vibrations (pulley attached with a spring)

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{110.8}{4 \times 0.352 + 0.007/3}}$$

f_1 =1.411 Hz (theoretical frequency for both small and big pulley)

Now we have both theoretical and natural frequencies for both big and small pulleys:

- Theoretical Frequency with both small and big pulley = 1.411 Hz
- Practical Frequency with small pulley = 1.777 Hz
- Practical Frequency with big pulley = 1.243 Hz
- Practical Frequency with only spring = 2.808 Hz

DISCUSSION

This experiment was performed to compare the theoretical and practical values of the natural frequency for the mass-pulley-spring system. The experiment was first done with the simple mass-spring system and the practical natural frequency for the system was 2.808 Hz [16]. Then this value was used to calculate the spring constant “k” of the spring. This value of k was used as the spring constant for calculating the theoretical natural frequency of the mass-spring-pulley system, which was calculated to be 1.411 Hz. The mass-spring-pulley system with the small pulley had the practical natural frequency of 1.777 Hz and with the big pulley had 1.243Hz [17].

The natural frequency for the small pulley was greater and the natural frequency for the big pulley was less than the theoretical natural frequency of the system. There was a significant difference between the frequencies of the big pulley and small pulley [18]. The practical frequencies should have been closer to the natural frequencies and the frequencies for the small and the big pulley should also have been close to each other, which is not the case. This must be due to the errors and assumptions we made during the experiment. The practical frequencies of both the pulleys should have been lesser than the theoretical frequencies, as the resistive forces were ignored while calculating the theoretical frequency [19].

There were certain assumptions made while calculating the theoretical frequency of the mass-spring-pulley system. It was assumed that:

1) The spring obeys Hooke's Law and is perfectly elastic, which is not the case in reality. The spring gets deformed slightly each time is stretched, but the deformation is very small and hence cannot be observed by human eye.[2][5]
2) The masses of the pulleys were considered negligible. This assumption has also led to the difference in the values of our frequencies.
3) The resistive forces were neglected. In reality there was air resistance which led to damping in the oscillations and hence the amplitude of the oscillations was reduced. If there were no resistive forces, the object would have never come back to rest.[5]
4) The friction between the pulley and the cable is negligible. This friction force also led to the difference in the practical values.
5) The motion of the object is in one direction i.e. it had only one degree of freedom. In reality the experiment was conducted in a 3-D environment and hence had six degrees of freedom. This is also one of the major reasons, as the object was possibly accelerating in more than one direction.
6) The tension was considered to be constant throughout the cable, which is not the case in reality.

There were uncertainties in the values measured via weighing machine and stop watch. The uncertainty in the stop watch was 0.01s and the weighing machine was 0.001 kg. The calculated uncertainty in the value of k is ± 0.2N/m. The calculated uncertainty in the theoretical natural frequency of the mass-pulley-spring system is ± 0.104 Hz [20]. The calculated uncertainty in the practical natural frequency of big pulley is ± 0.131 Hz and small pulley is ± 0.092 Hz. The percentage error in small pulley is 20.6% and the percentage error in the big pulley is 13.5% [21]. The errors which led to the changes in the practical and theoretical values are human errors. The human errors include:

1) The reaction time of a human, which always results in the errors in the values.
2) The start and the end point of the oscillations, which are possibly different.
3) The mass possibly would have been displaced at a certain angle to the horizontal. We assume that the mass was displaced downwards perpendicular to the horizontal.

Certain precautions could have been taken to minimize the errors:

1) Closing all the windows, in order to minimize the air resistance as much as possible.
2) A set square could have been used in order to make sure that the mass is displaced perpendicular to the horizontal.
3) Motion sensor could have been used in order to remove the reaction time from our measurements.
4) A pencil with a sharp tip could have been used as the reference point. This will fix the start and the end point of the oscillations at the equilibrium position and will make it easier for the user to use the stop watch to measure time.
5) It would have been better if the stop watch was started at the point where the displacement of the object was the greatest from the equilibrium position. This is because at the max displacement of the object from its equilibrium position, the velocity is zero.

CONCLUSION

The theoretical values can never be equal to the practical values, as there are always certain assumptions made while calculating the theoretical values, which can also be seen from the results. In reality it is not necessary that the assumptions that were made are true. If the difference between the practical and theoretical frequencies is greater than 10%, then it can be said that the calculated values had a lot of errors. For our case if the practical frequencies were between 1.270 Hz and 1.552 Hz, then the experiment had limited errors. Our theoretical frequency for the big pulley (1.243 Hz) is very close to the 1.270 Hz; whereas the frequency of the small pulley (1.777 Hz) is much greater than 1.552 Hz; therefore we can conclude that the experiment done with big pulley had very limited errors and the experiment done with the small pulley had many errors.

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